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# Multivalued Operators and Fixed-Point Theorems in Banach Algebras II

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**Abstract**—In this paper, two multivalued versions of the well-known hybrid fixed-point theorem of Dhage [1] in Banach algebras are proved. As an application, an existence theorem for a certain differential inclusion in Banach algebras is proved. 

**Keywords**—Multivalued operator, Fixed-point theorem, Integral inclusion

## 1. INTRODUCTION

The hybrid fixed-point theory initiated by Krasnoselskii [2] constitutes the fourth important category of fixed-point theory in the subject of nonlinear functional analysis and has several interesting applications to nonlinear differential and integral equations. The hybrid fixed-point theory for the single-valued mappings is growing with a good pace, and at present there are several hybrid fixed-point theorems available in literature. But, the case with the hybrid fixed-point theory for multivalued mappings is quite different. To the best of our knowledge, the multivalued analogue of Krasnoselskii's fixed-point theorem is the only multivalued hybrid fixed-point theorem available in the literature, so far. See [3,4] and the references therein. Very recently, the present author has obtained the multivalued analogues of some hybrid fixed-point theorems of Dhage [5,6] in Banach algebras and discussed some of their applications to nonlinear differential and integral inclusions under suitable conditions. In the present paper, we shall prove two multivalued analogues of a hybrid fixed-point theorem of the present author [1] and apply one of them to differential inclusions in Banach algebras for proving the existence of solutions under the mixed Lipschitz and compactness conditions.

## 2. PRELIMINARIES

Before stating the main fixed-point theorems, we give some useful definitions and preliminaries that will be used in the sequel.

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DEFINITION 2.1. *A mapping $T : X \to X$ is called $\mathcal{D}$-Lipschitzian, if there exists a continuous and nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, such that*

$$\|Tx - Ty\| \le \phi(\|x - y\|), \tag{2.1}$$

*for all $x, y \in X$, where $\phi(0) = 0$.*

Sometimes, we call for the function $\phi$ to be a *$\mathcal{D}$-function* of $T$ on $X$. Obviously, every Lipschitzian mapping is $\mathcal{D}$-Lipschitzian, but the converse may not be true.

Let $X$ be a Banach space and let $T \to X$. Then, $T$ is called a compact operator, if $\overline{T(X)}$ is a compact subset of $X$. $T : X \to X$ is called totally bounded, if, for any bounded subset $S$ of $X$, $T(S)$ is a totally bounded subset of $X$. Further, $T$ is called completely continuous, if it is continuous and totally bounded. Note that every compact operator is totally bounded, but the converse may not be true. However, the two notions are equivalent on a bounded subset of $X$.

We shall be interested in the multivalued analogue of the following hybrid fixed-point theorem of the present author involving the product of two operators in Banach algebras.

THEOREM 2.1. *(See [1].) Let $X$ be a Banach algebra and let $A, B : X \to X$ be two operators, such that*

(a) *$A$ is Lipschitzian with a Lipschitz constant $\alpha$,*
(b) *$B$ is compact and continuous, and*
(c) *$\alpha M < 1$, where $M = \sup\{\|Bx\| : x \in X\}$.*

*Then, either*

(i) *the operator equation $AxBx = x$ has a solution, or*
(ii) *the set $\mathcal{E} = \{u \in X \mid \lambda A(u/\lambda) Bu = u,\ 0 < \lambda < 1\}$ is unbounded.*

Note that the above fixed-point theorem involves the hypothesis of the continuity of the operator $T$, however, in the case of multivalued operators, we have the different types of continuities, namely, lower semicontinuity and upper semicontinuity. Here, in this present work, we shall formulate the fixed-point theorems, for each of these continuity criteria. Below, we give some preliminaries of the multivalued analysis, which will be needed in the sequel.

Let $X$ be a Banach space and let $P(X)$ denote the class of all subsets of $X$. Denote

$$P_f(X) = \{A \subset X \mid A \text{ is nonempty and has a property } f\}.$$

Thus, $P_{\mathrm{bd}}(X)$, $P_{\mathrm{cl}}(X)$, $P_{\mathrm{cv}}(X)$, $P_{\mathrm{cp}}(X)$, $P_{\mathrm{cl,bd}}(X)$, and $P_{\mathrm{cp,cv}}(X)$ denote the classes of all bounded, closed, convex, compact, closed-bounded and compact-convex subsets of $X$, respectively. Similarly, $P_{\mathrm{cl,cv,bd}}(X)$ and $P_{\mathrm{cp,cv}}(X)$ denote the classes of closed, convex and bounded and compact, convex subsets of $X$, respectively. A correspondence $T : X \to P_f(X)$ is called a multivalued operator or multivalued mapping on $X$. A point $u \in X$ is called a fixed-point of $T$, if $u \in Tu$. The multivalued operator $T$ is called lower semicontinuous (in short, l.s.c.), if $G$ is any open subset of $X$, then,

$$T^{-1(w)}(G) = \{x \in X \mid Tx \cap G \ne \emptyset\}$$

is an open subset of $X$. Similarly, the multivalued operator $T$ is called upper semicontinuous (in short, u.s.c.), if the set,

$$T^{-1}(G) = \{x \in X \mid Tx \subset G\},$$

is open in $X$ for every open set $G$ in $X$. Finally, $T$ is called continuous, if it is lower as well as upper semicontinuous on $X$. A multivalued map $T : X \to P_{\mathrm{cp}}(X)$ is called compact, if $\overline{T(X)}$ is a compact subset of $X$. $T$ is called totally bounded, if, for any bounded subset $S$ of $X$, $T(S) = \bigcup_{x \in S} Tx$ is a totally bounded subset of $X$. It is clear that every compact multivalued operator is totally bounded, but the converse may not be true. However, the two notions are equivalent on a bounded subset of $X$. Finally, $T$ is called completely continuous, if it is upper semicontinuous and totally bounded on $X$.

For any $A, B \in P_f(X)$, let us denote

$$\begin{aligned} A \pm B &= \{a \pm b \mid a \in A, b \in B\}, \\ A \cdot B &= \{ab \mid a \in A, b \in B\}, \\ \lambda A &= \{\lambda a \mid a \in A\}, \end{aligned}$$

for $\lambda \in \mathbb{R}$. Similarly, denote

$$|A| = \{|a| \mid a \in A\}$$

and

$$\|A\| = \sup\{|a| \mid a \in A\}.$$

Let $A, B \in P_{\mathrm{cl,bd}}(X)$ and let $a \in A$. Then, by

$$D(a, B) = \inf\{\|a - b\| \mid b \in B\}$$

and

$$\rho(A, B) = \sup\{D(a, B) \mid a \in A\}.$$

The function $H : P_{\mathrm{cl,bd}}(X) \times P_{\mathrm{cl,bd}}(X) \to \mathbb{R}^+$, defined by

$$H(A, B) = \max\{\rho(A, B), \rho(B, A)\},$$

is metric and is called the Hausdorff metric on $X$. It is clear that

$$H(0, C) = \|C\| = \sup\{\|c\| \mid c \in C\},$$

for any $C \in P_{\mathrm{cl,bd}}(X)$.

DEFINITION 2.2. *Let $T : X \to P_{\mathrm{bd,cl}}(X)$ be a multivalued operator. Then, $T$ is called a multivalued contraction, if there exists a constant $k \in (0, 1)$, such that, for each $x, y \in X$, we have*

$$H(T(x), T(y)) \leq k\|x - y\|.$$

*The constant $k$ is called a contraction constant of $T$.*

The following fixed-point theorem for multivalued contraction mappings appears in [7].

THEOREM 2.2. *Let $(X, d)$ be a complete metric space and let $T : X \to P_{\mathrm{bd,cl}}(X)$ be a multivalued contraction. Then, $T$ has a fixed point.*

## 3. MULTIVALUED FIXED-POINT THEORY

Before going to the main fixed-point results, we state some lemmas useful in the sequel.

LEMMA 3.1. *(See [8].) Let $(X, d)$ be a complete metric space and $T_1, T_2 : X \to P_{\mathrm{bd,cl}}(X)$ be two multivalued contractions with the same contraction constant $k$. Then,*

$$\rho(\mathrm{Fix}(T_1), \mathrm{Fix}(T_2)) \leq \frac{1}{1-k} \sup_{x \in X} \rho(T_1(x), T_2(x)).$$

LEMMA 3.2. *If $A, B \in P_{bd,cl}(X)$, then, $H(AC, BC) \leq H(0, C)H(A, B)$*

PROOF. The proof appears in [9], but for the sake of completeness, we give the details of it. Let $x \in AC$ and $y \in BC$ be arbitrary. Then, there exist $a \in A$, $b \in B$, and $c_1, c_2 \in C$, such that $x = ac_1$ and $y = bc_2$. Now,

$$\begin{aligned} D(x, BC) &= \inf\{\|x - y\| \mid y \in BC\} \\ &= \inf\{\|x - bc_2\| \mid b \in B, c_2 \in C_2\} \\ &= \inf\{\|ac_1 - bc_2\| \mid b \in B, c_2 \in C_2\} \\ &\leq \inf\{\|ac_1 - bc_1\| + \|bc_1 - bc_2\| \mid b \in B, c_2 \in C_2\} \\ &\leq \inf\{\|a - b\| \, \|c_1\| + \|b\| \, \|c_1 - c_2\| \mid b \in B, c_2 \in C_2\} \\ &= \inf\{\|a - b\| \, \|c_1\| \mid b \in B\} \\ &= D(a, B) \|c_1\|. \end{aligned}$$

Again,

$$\begin{aligned} \rho(AC, BC) &= \sup\{D(x, BC) \mid x \in A\} \\ &= \sup\{D(a, B) \|c_1\| \mid a \in A, c_1 \in C\} \\ &\leq \sup\{D(a, B) \|C\| \mid a \in A\} \\ &= \rho(A, B) \|C\| \\ &= \rho(A, B) H(0, C). \end{aligned}$$

Similarly,

$$\rho(BC, AC) = \rho(B, A) H(0, C).$$

Hence,

$$\begin{aligned} H(AC, BC) &= \max\{\rho(AC, BC), \rho(BC, AC)\} \\ &\leq \max\{\rho(A, B) H(0, C), \rho(B, A) H(0, C)\} \\ &= H(0, C) \max\{\rho(A, B), \rho(B, A)\} \\ &= H(0, C) H(A, B). \end{aligned}$$

The proof of the lemma is complete. ∎

Now, we state two key results, which are useful in the sequel.

THEOREM 3.1. *(See [10, p. 124].) Let $X$ be a Banach space and let $T : X \to X$ be completely continuous. Then, either*

(i) *the operator equation $x = Tx$ has a solution, or*
(ii) *the set $\mathcal{E} = \{u \in X \mid u = \lambda Tu, \ 0 < \lambda < 1\}$ is unbounded.*

THEOREM 3.2. *(See [11].) Let $S$ be a nonempty and closed subset of a Banach space $X$ and let $Y$ be a metric space. Assume that the multivalued operator $F : S \times Y \to P_{cl,cv}(S)$ be a multivalued mapping satisfying*

(a) *$H(F(x_1, y), F(x_2, y)) \leq k\|x_1 - x_2\|$, for each $(x_1, y), (x_2, y) \in S \times Y$,*
(b) *for every $x \in S$, $F(x, \cdot)$ is lower semicontinuous (briefly, l.s.c.) on $Y$.*

*Then, there exists a continuous mapping $f : S \times Y \to S$, such that $f(x, y) \in F(f(x, y), y)$, for each $(x, y) \in S \times Y$.*

THEOREM 3.3. *Let $X$ be a Banach space and let $A : X \to P_{\mathrm{bd,cl,cv}}(X)$, $B : X \to P_{\mathrm{cp,cv}}(X)$ be two multivalued operators, such that*

(a) *$A$ is multivalued contraction with a contraction $k$,*
(b) *$B$ is l.s.c. and compact,*
(c) *$AxBy$ is a convex subset of $S$, for each $x, y \in S$, and*
(d) *$Mk < 1$, where $M = \|B(X)\| = \sup\{\|B(x)\| \mid x \in X\}$.*

*Then, either*

(i) *the operator inclusion $x \in AxBx$ has a solution, or*
(ii) *the set $\mathcal{E} = \{u \in X \mid \lambda u \in A(\lambda u)Bu, \lambda > 1\}$ is unbounded.*

PROOF. Define a multivalued operator $T : X \times X \to P_{\mathrm{cp,cv}}(X)$ by

$$T(x,y) = AxBy,$$

for $x \in X$ and $y \in X$. We show that $T(x,y)$ is multivalued contraction in $x$, for each fixed $y \in X$. Let $x_1, x_2 \in X$ be arbitrary. Then, by Lemma 3.2,

$$\begin{aligned} H(T(x_1,y), T(x_2,y)) &= H(A(x_1)B(y), A(x_2)B(y)) \\ &\le H(A(x_1), A(x_2))\, H(0, By) \\ &\le k\|x_1 - x_2\|\, \|B(S)\| \\ &\le kM\|x_1 - x_2\|. \end{aligned}$$

This shows that the multivalued operator $T_y(\cdot) = T(.,y)$ is a contraction on X with a contraction constant $kM$. Hence, an application of Covitz-Nadler fixed-point theorem yields that the fixed-point set,

$$\mathrm{Fix}(T_y) = \{x \in X \mid x \in A(x)B(y)\},$$

is nonempty and closed subset of $S$ for each $y \in X$.

Now, the operator $T(x,y)$ satisfies all the conditions of Theorem 3.2 and hence, an application of it yields that there exists a continuous mapping $f : X \times X \to X$, such that $f(x,y) \in A(f(x,y)) + B(y)$. Let us define $C(y) = \mathrm{Fix}(T_y)$, $C : X \to P_{\mathrm{cl}}(X)$. Let us consider the single-valued operator $c : X \to X$ defined by $c(x) = f(x,x)$, for each $x \in X$. Then, $c$ is a continuous mapping having the property that

$$c(x) = f(x,x) \in A(f(x,x)) + B(x) = A(c(x)) + B(x),$$

for each $x \in X$.

Now, we will prove that $c$ is totally bounded on $X$. Let $S$ be a bounded subset of $X$. Then, there is a real number $r > 0$, such that $\|x\| \le r$, for all $x \in S$. To finish, it is sufficient to show that $C$ is compact on $S$. Let $\epsilon > 0$. Since $B$ is compact on $X$ and $B(S) \subset B(X)$, we have that $B(S)$ is compact. Then, there exists $Y = \{y_1, \dots, y_n\} \subset X$, such that

$$B(S) \subset \{w_1, \dots, w_n\} + \mathcal{B}\left(0, \frac{1 - Mk}{\delta}\epsilon\right) \subset \bigcup_{i=1}^{n} B(y_i) + \mathcal{B}\left(0, \frac{1 - Mk}{\delta}\epsilon\right),$$

where $w_i \in B(y_i)$, for each $i = 1, 2, \dots, n$; and $\mathcal{B}(a,r)$ is an open ball in $X$ centered at $a \in X$ of radius $r$. By hypothesis $(a)$,

$$\begin{aligned} \|Ax\| &\le \|A0\| + H(A0, Ax) \\ &\le \|A0\| + k\|x\| \\ &\le \delta, \end{aligned}$$

for all $x \in S$, where

$$\delta = \|A0\| + kr < \infty.$$

Hence, $A(S)$ is a bounded subset of $X$ with bound $\delta$.

Therefore, for each $y \in S$, it follows that,

$$B(y) \subset \bigcup_{i=1}^{n} B(y_i) + \mathcal{B}\left(0, \frac{1 - Mk}{\delta}\epsilon\right),$$

and hence, there exists an element $y_k \in Y$, such that

$$\rho(B(y), B(y_k)) < \frac{1 - Mk}{\delta}\epsilon.$$

Then,

$$\begin{aligned}
\rho(C(y), C(y_k)) &= \rho(\mathrm{Fix}(T_y), \mathrm{Fix}(T_{y_k})) \\
&\leq \frac{1}{1 - Mk} \sup_{x \in S} \rho(T_y(x), T_{y_k}(x)) \\
&= \frac{1}{1 - Mk} \sup_{x \in Y} \rho(A(x)B(y), A(x)B(y_k)) \\
&\leq \frac{1}{1 - Mk} \sup_{x \in Y} \rho(0, Ax)\,\rho(B(y), B(y_k)) \\
&< \frac{\delta}{(1 - Mk)} \frac{(1 - Mk)}{\delta}\epsilon \\
&= \epsilon.
\end{aligned}$$

It follows that, for each $u \in C(y)$, there is $v_k \in C(y_k)$, such that $\|u - v_k\| < \epsilon$. Hence, for each $y \in S$, $C(y) \subset \bigcup_1^n \mathcal{B}(v_i, \epsilon)$, where $v_i \in C(y_i)$, $i = 1, 2, \dots, n$. Therefore, $c(S) \subset C(S) \subset \bigcup_1^n \mathcal{B}(v_i, \epsilon)$ and so $c(S)$ is a totally bounded set in $X$. Thus, $c$ is completely continuous operator on $X$.

Finally, note that the mapping $c : X \to X$ satisfies all the assumptions of Schaefer's fixed-point theorem and hence, an application of it yields that, either

(i) the operator equation $x = cx$ has a solution, or

(ii) the set $\mathcal{E} = \{u \in X \mid u = \lambda cu,\ 0 < \lambda < 1\}$ is unbounded.

This, by definition of $c$, further implies that either

(i) the operator equation $x \in AxBx$ has a solution, or

(ii) the set $\mathcal{E} = \{u \in X \mid \lambda u \in A(cu)Bu = A(\lambda u)Bu,\ \lambda > 1\}$ is unbounded.

This completes the proof. ∎

A Hausdorff measure of noncompactness $\chi$ of a bounded set $S$ in $X$ is a nonnegative real number $\chi(S)$ defined by

$$\chi(S) = \inf\left\{r > 0 : A \subset \bigcup_{i=1}^{n} \mathcal{B}(x_i, r),\ x_i \in X\right\}.$$

The function $\chi$ enjoys the following properties.

($\chi_1$) $\chi(S) = 0 \Leftrightarrow S$ is precompact.

($\chi_2$) $\chi(S) = \chi(\bar{S}) = \chi(\overline{\mathrm{co}}\, S)$, where $\bar{S}$ and $\overline{\mathrm{co}}\, S$ denote, respectively, the closure and the closed convex hull of $S$.

($\chi_3$) $S_1 \subset S_2 \Rightarrow \chi(S_1) \leq \chi(S_2)$

($\chi_4$) $\chi(S_1 \cup S_2) = \max\{\chi(S_1), \chi(S_2)\}$.

($\chi_5$) $\chi(\lambda S) = |\lambda|\chi(S), \forall\, \lambda \in \mathbb{R}$.

($\chi_6$) $\alpha(S_1 + S_2) \leq \chi(S_1) + \chi(S_2)$.

The details of measures of noncompactness and their properties appear in [12,13].

DEFINITION 3.1. *A mapping $T : X \to X$ is called condensing, if, for any bounded subset $S$ of $X$, $T(S)$, is bounded and*

$$\chi(T(S)) < \chi(A), \qquad \chi(S) > 0.$$

Note that contraction and completely continuous mappings are condensing, but the converse may not be true. Some interesting results concerning the condensing mappings and fixed points appear in [14,15]. The following fixed-point theorem for condensing multivalued mappings is well known.

THEOREM 3.4. *(See [16].) Let $X$ be a Banach space and let $T : X \to P_{\mathrm{cp,cv}}(X)$ be a upper semicontinuous and $\chi$-condensing multivalued operator. Then, $T$ has a fixed-point point.*

We need the following result, which is useful in the sequel.

LEMMA 3.3. *(See [17].) If $S_1, S_2 \in P_{\mathrm{bd}}(X)$, then,*

$$\chi(S_1.S_2) \leq \chi(S_1)\|S_2\| + \chi(S_2)\|S_1\|.$$

THEOREM 3.5. *Let $X$ be a Banach algebra and let $A : X \to P_{\mathrm{bd,cl,cv}}(X)$, $B : X \to P_{\mathrm{cl,cv}}(X)$ be two multivalued operators, satisfying*

(a) *$A$ is Lipschitzian with a Lipschitz constant $k$,*
(b) *$B$ is compact and upper semi-continuous,*
(c) *$AxBx$ is a convex subset of $X$, for each $x \in X$, and*
(d) *$M\phi(r) < r$, whenever $r > 0$, with $M = \|B(X)\|$.*

*Then, either*

(i) *the operator inclusion $x \in AxBx$ has a solution, or*
(ii) *the set $\mathcal{E} = \{u \in X \mid \lambda u \in AuBu,\ \lambda > 1\}$ is unbounded.*

PROOF. Define a mapping $T : X \to P_{\mathrm{cl}}(X)$ by

$$Tx = AxBx, \qquad x \in X. \tag{3.5}$$

We shall show that $T$ satisfies all the conditions of Theorem 2.2 on $X$.

STEP I. First, we claim that $T$ defines a multivalued map $T : X \to P_{\mathrm{cp,cv}}(X)$. Obviously, $Tx$ is convex subset of $X$, for each $x \in X$. Next, from Lemma 3.3, it follows that

$$\chi(Tx) = \chi(Ax \cdot Bx) \leq \chi(Ax) . \|B(x)\| + \chi(Bx) . \|A(x)\| = 0,$$

for every $x \in X$ and the claim follows.

STEP II. Now, we shall show that mapping $T$ is an upper semicontinuous on $X$. Let $\{x_n\}$ be a sequence in $X$ converging to the point $x^* \in X$ and let $\{y_n\}$ be sequence defined by $y_n \in Tx_n$ converging to the point $y^*$. It is enough to prove that $y^* \in Tx^*$. Now, for any $x, y \in X$, we have

$$\begin{aligned} H(Tx, Ty) &= H(AxBx, AyBy) \\ &\leq H(AxBx, AyBx) + H(AyBx, AyBy) \\ &\leq H(Ax, Ay)\, H(0, Bx) + H(0, Ay)\, H(Bx, By) \\ &\leq k\|x - y\|\,\|B(X)\| + \|Ay\|\ H(Bx, By) \\ &\leq Mk\|x - y\| + \|Ay\|\ H(Bx, By). \end{aligned} \tag{3.6}$$

Since $B$ is u.s.c., it is $H$-upper semicontinuous and consequently,

$$H(Bx_n, Bx^*) \to 0, \quad \text{whenever } x_n \to x^*.$$

Therefore,

$$\begin{aligned}
D(y^*, Tx^*) &\leq \lim_{n\to\infty} D(y_n, Tx^*)\\
&\leq H(Tx_n, Tx^*)\\
&\leq Mk\,\|x_n - x^*\| + \|Ay^*\|\; H(Bx_n, Bx^*)\\
&\longrightarrow 0, \qquad \text{as } n \to \infty.
\end{aligned}$$

This shows that the multivalued operator $T$ is an upper semicontinuous on $X$.

STEP III. Finally, we show that $T$ is $\chi$-condensing on $X$. Let $S$ is a bounded subset of $X$. Then, there is a real number $r > 0$, such that $\|x\| \leq r$, for all $x \in S$. Then, we have the following estimate concerning the multivalued operators $A$ and $B$. Now, it can be shown, as in the proof of Theorem 3.1, that $A(S)$ is a bounded subset of $X$ with bound $\delta$, where $\delta$ is given by (3.3). Since $B$ is compact, $B(X)$ is a precompact subset of $X$. Let $\epsilon > 0$ be given. Then, there exists $Z = \{x_1, \dots, x_n\}$ in $X$, such that

$$\begin{aligned}
B(X) &\subset \bigcup_{i=1}^{n} \mathcal{B}\left(y_i, \frac{\epsilon}{\delta}\right)\\
&\subset \{y_1, \dots, y_n\} + \mathcal{B}\left(0, \frac{\epsilon}{\delta}\right)\\
&\subset \bigcup_{i=1}^{n} \mathcal{B}\left(Bx_i, \frac{\epsilon}{\delta}\right),
\end{aligned}$$

for some $y_i \in Bx_i$, for $i = 1, \dots, n$. Therefore, for every $x \in S$, there exists an $x_i \in Z$, such that

$$\rho(Bx, Bx_i) < \frac{\epsilon}{\delta}.$$

Let $\chi(S) = r$. Then, we have

$$S \subset \bigcup_{i=1}^{m} \mathcal{B}(x_i, r + \epsilon).$$

Now, for any $x \in S$, we have

$$\begin{aligned}
\rho(Tx, Tx_i) &\leq H(Tx, Tx_i)\\
&\leq Mk\,\|x - x_i\| + \delta H(Bx, Bx_i)\\
&< Mk\,\|x - x_i\| + \delta\,\frac{\epsilon}{\delta}\\
&\leq Mk\,(r + \epsilon) + \epsilon,
\end{aligned}$$

for each $i =, \dots, n$. Again, each $Tx_i$ is compact, for each $i =, \dots, n$, there are $y_1^i, \dots, y_{n(i)}^i$ in $Tx_i$, such that

$$Tx_i \subset \bigcup_{j=1}^{n(i)} \mathcal{B}\left(y_j^i, \frac{\epsilon}{2}\right).$$

Now, from (3.6), it follows that

$$T(S) \subset \bigcup_{i=1}^{n} \left\{ \bigcup_{j=1}^{n(i)} \left\{\mathcal{B}\left(y_j^i, Mk\,(r+\epsilon)+\epsilon\right)\right\} \right\}.$$

Therefore,

$$\chi(T(S)) < Mk\,(r+\epsilon) + \epsilon.$$

Since $\epsilon$ is arbitrary, one has

$$\chi(T(S)) \leq Mkr = Mk\chi(S) < \chi(S),$$

whenever $\chi(S) > 0$. This shows that $T$ is $\chi$-condensing on $S$ into itself. Now, an application of Theorem 3.4 yields that, either

(i) the operator inclusion $x \in Tx$ has a solution, or
(ii) the set $\mathcal{E} = \{u \in X \mid \lambda u \in Tu,\ \lambda > 1\}$ is unbounded.

This, by definition of $T$, further implies that

(i) the operator inclusion $x \in AxBx$ has a solution, or
(ii) the set $\mathcal{E} = \{u \in X \mid \lambda u \in AuBu,\ \lambda > 1\}$ is unbounded.

This completes the proof. ∎

## 4. DIFFERENTIAL INCLUSIONS

In this section, we prove the existence theorems for the differential inclusions in Banach algebras by the applications of the abstract results of the previous section under generalized Lipschitz and Carathéodory conditions.

Given a closed and bounded interval $J = [0, a]$ in $\mathbb{R}$, for some $a \in \mathbb{R}$, $a > 0$, consider the differential inclusion (in short, DI)

$$\begin{gathered}\left(\frac{x(t)}{f(t, x(t))}\right)' \in G(t, x(t)), \qquad \text{a.e., } t \in J, \\ x(0) = x_0 \in \mathbb{R},\end{gathered} \tag{4.1}$$

where $f : J \times \mathbb{R} \to \mathbb{R} - \{0\}$ is continuous and $G : J \times \mathbb{R} \to P_{\mathrm{cp,cv}}(\mathbb{R})$.

By a solution to DI (4.1), we mean a function $x \in AC(J, \mathbb{R})$ that satisfies

$$\left(\frac{x(t)}{f(t, x(t))}\right)' = v(t), \qquad t \in J,$$

for some $v \in L^1(J, \mathbb{R})$, satisfying $v(t) \in G(t, x(t))$ a.e. $t \in J$, where $AC(J, \mathbb{R})$ is the space of all absolutely continuous real valued functions on $J$.

The DI (4.1) is new to the theory of differential inclusions and the special cases of it have been discussed in the literature extensively. For example, if $f(t, x) = 1$, then, the DI (4.1) reduces to DI

$$\begin{gathered}x' \in G(t, x), \qquad \text{a.e., } t \in J \\ x(0) = x_0 \in \mathbb{R}.\end{gathered}$$

There is a considerable work available in the literature for the DI (4.2). See [12,18–20], etc. Similarly, in the special case when $G(t, x) = \{g(t, x)\}$, we obtain the differential equation,

$$\begin{gathered}\left(\frac{x(t)}{f(t, x(t))}\right)' = g(t, x(t)), \qquad \text{a.e., } t \in J \\ x(0) = x_0 \in \mathbb{R}.\end{gathered} \tag{4.3}$$

The differential equation (4.3) has been studied recently in [21,22], for the existence of solutions. Therefore, it is of interest to discuss DI (4.3) for various aspects of its solution under suitable conditions. In this section, we shall prove the existence of the solutions of DI (4.1) in the space $c(J, \mathbb{R})$ of continuous real-valued functions on $J$, under the mixed generalized Lipschitz and Carathéodory conditions.

Define a norm $\|\cdot\|$ in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)|.$$

Again, define a multiplication "$\cdot$" by

$$(x.y)(t) = x(t)\,y(t), \qquad \forall\, t \in J.$$

Then, $C(J,\mathbb{R})$ is a Banach algebra with the above norm and multiplication in it.

We need the following definitions in the sequel.

DEFINITION 4.1. *A multivalued map $F : J \to P_f(\mathbb{R})$ is said to be measurable, if, for any $y \in X$, the function $t \to d(y, F(t)) = \inf\{|y-x| : x \in F(t)\}$ is measurable.*

DEFINITION 4.2. *A measurable multivalued function $F : J \to P_{\mathrm{cp}}(\mathbb{R})$ is said to be integrably bounded, if there exists a function $h \in L^1(J,\mathbb{R})$, such that $\|v\| \le h(t)$, a.e. $t \in J$, for all $v \in F(t)$.*

REMARK 4.1. It is known that, if $F : J \to P_f(\mathbb{R})$ is an integrably bounded multifunction, then, the set $S_F^1$ of all Lebesgue integrable selections of $F$ is closed and nonempty. See [7].

DEFINITION 4.3. *A multivalued function $\beta : J \times \mathbb{R} \to P_{\mathrm{bd,cl}}(\mathbb{R})$ is called Carathéodory if*

(i) *$t \mapsto \beta(t,x)$ is measurable, for each $x \in E$, and*
(ii) *$x \mapsto \beta(t,x)$ is an upper semicontinuous almost everywhere, for $t \in J$.*

DEFINITION 4.4. *A Carathéodory multifunction $\beta(t,x)$ is called $L_X^1$-Carathéodory, if there exists a function $h \in L^1(J,\mathbb{R})$, such that*

$$\|\beta(t,x)\| \le h(t), \qquad \text{a.e., } t \in J,$$

*for all $x \in \mathbb{R}$.*

Denote

$$S_\beta^1(x) = \left\{ v \in L^1(J,E) \,\middle|\, v(t) \in \beta(t, x(t)),\ \text{a.e., } t \in J \right\}.$$

Then, we have the following lemmas due to [23].

LEMMA 4.1. *Let $E$ be a Banach space. If $\dim(E) < \infty$ and $\beta : J \times E \to P_{\mathrm{bd,cl}}(E)$ is $L^1$-Carathéodory, then, $S_\beta^1(x) \neq \emptyset$, for each $x \in E$.*

LEMMA 4.2. *Let $E$ be a Banach space, $\beta$ a Carathéodory multimap with $S_\beta^1 \neq \emptyset$ and let $\mathcal{L} : L^1(J,E) \to C(J,E)$ be a continuous linear mapping. Then, the operator,*

$$\mathcal{L} \circ S_\beta^1 : C(J,E) \to P_{\mathrm{bd,cl}}(C(J,E))$$

*is a closed graph operator on $C(J,E) \times C(J,E)$.*

We consider the following hypotheses in the sequel.

($\mathrm{H}_1$) The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a bounded function $\ell : J \to \mathbb{R}$ with bound $\|\ell\|$, satisfying

$$|f(t,x) - f(t,y)| \le \ell(t)\,|x-y|, \qquad \text{a.e., } t \in J,$$

for all $x, y \in \mathbb{R}$.

($\mathrm{H}_2$) The multifunction $G : J \times \mathbb{R} \to P_{\mathrm{cp,cv}}(\mathbb{R})$ is $L_X^1$-Carathéodory.

($\mathrm{H}_3$) There exists a function $\gamma \in L^1(J,\mathbb{R})$ with $\gamma(t) > 0$, a.e. $t \in J$ and a nondecreasing function $\Omega : \mathbb{R}^+ \to (0,\infty)$, such that

$$\|G(t,x)\| \le \gamma(t)\,\Omega(|x|), \qquad \text{a.e., } t \in J,$$

for all $x \in \mathbb{R}$.

THEOREM 4.1. *Assume that the hypotheses ($H_1$)–($H_3$) hold. Further, if*

$$\int_{C_1}^{\infty} \frac{ds}{\Omega(s)} > C_2 \|\gamma\|_{L^1},$$

*where*

$$C_1 = \frac{F\,|x_0/f(0,x_0)|}{1 - \|\ell\|\left(|x_0/f(0,x_0)| + \|h\|_{L^1}\right)},$$

$$C_1 = \frac{F}{1 - \|\ell\|\left(|x_0/f(0,x_0)| + \|h\|_{L^1}\right)},$$

*and*

$$\|\ell\|\left(\left|\frac{x_0}{f(0,x_0)}\right| + \|h\|_{L^1}\right) < 1,$$

*then, DI (4.1) has a solution on $J$.*

PROOF. Let $X = C(J,\mathbb{R})$ and consider the two multivalued mappings $A$ and $B$ on $X$, defined by

$$Ax(t) = f(t, x(t)) \tag{4.5}$$

and

$$Bx(t) = \left\{u \in X \mid u(t) = \frac{x_0}{f(0,x_0)} + \int_0^t v(s)\,ds,\ v \in S_G^1(x)\right\}, \tag{4.6}$$

for all $t \in J$.

Then, DI (4.1) is equivalent to the operator inclusion,

$$x(t) \in Ax(t)\,Bx(t), \qquad t \in J.$$

We shall show that the multivalued operators $A$ and $B$ satisfy all the conditions of Theorem 3.3. Clearly, the operator $B$ is well defined since $S_G^1(x) \neq \emptyset$, for each $x \in X$.

STEP I. We first show that the operators $A$ and $B$ define the multivalued operators $A, B : X \to P_{\mathrm{cp,cv}}(X)$. The case of $A$ is obvious since it is a single-valued operator on $S$. We only prove the claim for the operator $B$. Let $\{u_n\}$ be a sequence in $Bx$ converging to a point $u$. Then, there is a sequence $\{v_n\} \subset S_G^1(x)$, such that

$$u_n(t) = \frac{x_0}{f(0,x_0)} + \int_0^t v_n(s)\,ds$$

and $v_n \to v$. Since $G(t,x)$ is closed, for each $(t,x) \in J \times \mathbb{R}$, we have $v \in S_G^1(x)$. As a result,

$$u(t) = \frac{x_0}{f(0,x_0)} + \int_0^t v(s)\,ds \in Bx(t), \qquad \forall t \in J.$$

Hence, $B$ has closed values on $X$. Again, let $u_1, u_2 \in Ax$. Then, the are $v_1, v_2 \in S_G^1(x)$, such that

$$u_1(t) = \frac{x_0}{f(0,x_0)} + \int_0^t v_1(s)\,ds, \qquad t \in J,$$

and

$$u_2(t) = \frac{x_0}{f(0,x_0)} + \int_0^t v_2(s)\,ds, \qquad t \in J.$$

Now, for any $\gamma \in [0,1]$,

$$\begin{aligned}
\gamma u_1(t) + (1-\gamma) u_2(t) &= \gamma \left( \frac{x_0}{f(0,x_0)} + \int_0^t v_1(s)\, ds \right) \\
&\quad + (1-\gamma) \left( \frac{x_0}{f(0,x_0)} + \int_0^t v_2(s)\, ds \right) \\
&= \frac{x_0}{f(0,x_0)} + \int_0^t [\gamma v_1(s) + (1-\gamma) v_2(s)]\, ds \\
&= \frac{x_0}{f(0,x_0)} + \int_0^t v(s)\, ds,
\end{aligned}$$

where $v(t) = \gamma v_1(t) + (1-\gamma)v_2(s) \in G(t, x(t))$, for all $t \in J$. Hence, $\gamma u_1 + (1-\gamma)u_2 \in Bx$ and consequently, $Bx$ is convex, for each $x \in X$. As a result, $A$ defines a multivalued operator $B : X \to P_{\mathrm{bd,cl,cv}}(X)$. Again, let $t, \tau \in J$. Then, for any $u \in Bx$, we have

$$\begin{aligned}
|u(t) - u(\tau)| &\le \left| \int_0^t v(s)\, ds - \int_0^\tau v(s)\, ds \right| \\
&\le \left| \int_\tau^t \right| v(s) \,|\, ds| \\
&\le |p(t) - p(\tau)|,
\end{aligned}$$

where $p(t) = \int_0^t h(s)\, ds$.

Since $p$ is continuous on compact interval $J$, it is uniformly continuous. Hence, $Bx$ is compact by Arzela-Ascoli theorem. Thus, we have $B : X \to P_{\mathrm{cp,cv}}(X)$. Hence, $A, B : X \to P_{\mathrm{cp,cv}}(X)$.

STEP II. To show $A$ a contraction on $X$, let $x, y \in X$. Then,

$$\begin{aligned}
\|Ax - Ay\| &= \sup_{t\in J} |Ax(t) - Ay(t)| \\
&= \sup_{t\in J} |f(t, x(t)) - f(t, y(t))| \\
&\le \sup_{t\in J} \ell(t)\, |x(t) - y(t)| \\
&\le \|\ell\|\, \|x - y\|,
\end{aligned}$$

showing that $A$ is a Lipschitzian on $X$.

STEP III. Next, we show that $B$ is compact and upper semicontinuous on $X$. First, we prove that $B(X)$ is totally bounded on $X$. To do this, it is enough to prove that $B(X)$ is a uniformly bounded and equicontinuous set in $X$. To see this, let $u \in B(X)$ be arbitrary. Then, there is a $v \in S_G^1(x)$, such that

$$u(t) = \int_0^t v(s)\, ds,$$

for some $x \in X$. Hence,

$$\begin{aligned}
|u(t)| &\le \int_0^t |v(s)|\, ds \\
&\le \int_0^t \|G(s, x(s))\|\, ds \\
&\le \int_0^t h(s)\, ds \\
&= \|h\|_{L^1},
\end{aligned}$$

for all $x \in S$ and so, $B(X)$ is a uniformly bounded set in $X$. Again, as in Step I, it is proved that

$$|u(t) - u(\tau)| \leq |p(t) - p(\tau)|,$$

where $p(t) = \int_0^t h(s)\, ds$.

Notice that $p$ is a continuous function on $J$, so, it is uniformly continuous on $J$. As a result, we have that

$$|u(t) - u(\tau)| \to 0, \qquad \text{as } t \to \tau.$$

This shows that $B(X)$ is a equicontinuous set in $X$. Next, we show that $B$ is an upper semicontinuous multivalued mapping on $X$. Let $\{x_n\}$ be a sequence in $X$, such that $x_n \to x_*$. Let $\{y_n\}$ be a sequence, such that $y_n \in Bx_n$ and $y_n \to y_*$. We shall show that $y_* \in Bx_*$. Since $y_n \in Bx_n$, there exists a $v_n \in S_G^1(x_n)$, such that

$$y_n(t) = \frac{x_0}{f(0, x_0)} + \int_0^t v_n(s)\, ds, \qquad t \in J.$$

We must prove that there is a $v_* \in S_G^1(x_*)$, such that

$$y_*(t) = \frac{x_0}{f(0, x_0)} + \int_0^t v_*(s)\, ds, \qquad t \in J.$$

Consider the continuous linear operator $\mathcal{K} : L^1(J, \mathbb{R}) \to C(J, E)$ defined by

$$\mathcal{K}y(t) = \int_0^t v(s)\, ds, \qquad t \in J.$$

Now, we have

$$\left\| \left( y_n - \frac{x_0}{f(0, x_0)} \right) - \left( y_* - \frac{x_0}{f(0, x_0)} \right) \right\| \to 0, \qquad \text{as } n \to 0.$$

From Lemma 4.2, it follows that $\mathcal{K} \circ S_G^1$ is a closed graph operator. Also, from the definition of $K$, we have

$$y_n(t) - \frac{x_0}{f(0, x_0)} \in \mathcal{K} \circ S_G^1(x_n).$$

Since $y_n \to y_*$, there is a point $v_* \in S_G^1(x_*)$, such that

$$y_*(t) = \frac{x_0}{f(0, x_0)} + \int_0^t v_*(s)\, ds, \qquad t \in J.$$

This shows that $B$ is a upper semicontinuous operator on $X$. Thus, $B$ is an upper semicontinuous and compact operator on $X$.

Step IV. Here, we show that $AxBx$ is a convex subset of $X$, for each $x \in X$. Let $x \in X$ be arbitrary and let $w, y \in X$. Then, there are $u, v \in S_G^1(x)$, such that

$$w = [f(t, x(t))] \left( \frac{x_0}{f(0, x_0)} + \int_0^t u(s)\, ds \right)$$

and

$$y = [f(t, x(t))] \left( \frac{x_0}{f(0, x_0)} + \int_0^t v(s)\, ds \right).$$

Now, for any $\lambda \in [0,1]$,

$$\begin{aligned}
\lambda y + (1-\lambda) w &= \lambda \left[f(t, x(t))\right] \left(\frac{x_0}{f(0,x_0)} + \int_0^t v(s)\, ds\right) \\
&\quad + (1-\lambda) \left[f(t, x(t))\right] \left(\frac{x_0}{f(0,x_0)} + \int_0^t v(s)\, ds\right) \\
&= \left[f(t, x(t))\right] \left(\lambda \frac{x_0}{f(0,x_0)} + \int_0^t \lambda v(s)\, ds\right) \\
&\quad + \left[f(t, x(t))\right] \left((1-\lambda) \frac{x_0}{f(0,x_0)} + \int_0^t (1-\lambda) v(s)\, ds\right) \\
&= \left[f(t, x(t))\right] \left(\frac{x_0}{f(0,x_0)} + \int_0^t \left[\lambda u(s) + (1-\lambda) v(s)\right] ds\right).
\end{aligned}$$

Since $G(t,x(t))$ is convex, $z = \lambda y + (1-\lambda) w \in G(t,x(t))$, for all $t \in J$ and so, $z \in X_G^1(x)$. As a result, $\lambda y + (1-\lambda)w \in AxBx$. Hence, $AxBx$ is a convex subset of $X$.

STEP V. Finally, from condition (4.1), it follows that

$$Mk = \|\ell\| \left(\left|\frac{x_0}{f(0,x_0)}\right| + \|h\|_{L^1}\right) < 1.$$

Thus, all the conditions of Theorem 2.1 are satisfied and hence, a direct application of it yields that either Conclusion (i) or Conclusion (ii) holds. We show that Conclusion (ii) is not possible. Let $u \in \mathcal{E}$ be arbitrary. Then, we have, for any $\lambda > 1$,

$$\begin{aligned}
\lambda u(t) &\in Au(t)\, Bu(t) \\
&= \left[f(t, u(t)] \left(\frac{x_0}{f(0,x_0)} + \int_0^t G(s, u(s))\right) ds\right), \qquad t \in J,
\end{aligned}$$

for some real number $\lambda > 1$. Therefore,

$$\lambda u(t) \in \left[f(t, u(t))\right] \left(\frac{x_0}{f(0,x_0)} + \int_0^t G(s, u(s))\, ds\right)$$

or

$$u(t) = \lambda^{-1} \left[f(t, u(t))\right] \left(\frac{x_0}{f(0,x_0)} + \int_0^t v(s)\, ds\right),$$

for some $v \in S_G^1(u)$.

Now,

$$\begin{aligned}
|u(t)| &= \left|\lambda^{-1} \left[f(t, u(t))\right]\right| \left|\left(\frac{x_0}{f(0,x_0)} + \int_0^t v(s)\, ds\right)\right| \\
&\le \left|\left[f(t, u(t))\right]\right| \left(\left|\frac{x_0}{f(0,x_0)}\right| + \int_0^t |v(s)|\, ds\right) \\
&\le \left[|f(t, u(t)) - f(t,0)| + |f(t,0)|\right] \\
&\quad \times \left(\left|\frac{x_0}{f(0,x_0)}\right| + \int_0^t \|G(s, u(s))\|\, ds\right) \\
&\le |\ell(t)|\, |u(t)| \left(\left|\frac{x_0}{f(0,x_0)}\right| + \int_0^t \|G(s, u(s))\|\, ds\right) \\
&\quad + F\left(\left|\frac{x_0}{f(0,x_0)}\right| + \int_0^t \|G(s, u(s))\|\, ds\right) \\
&\le \|\ell\|\, |u(t)| \left(\left|\frac{x_0}{f(0,x_0)}\right| + \|h\|_{L^1}\right) \\
&\quad + F\left(\left|\frac{x_0}{f(0,x_0)}\right| + \int_0^t \phi(s)\, \Omega(|u(s)|)\, ds\right) \\
&= C_1 + C_2 \int_0^t \phi(s)\, \Omega(|u(s)|)\, ds,
\end{aligned}$$

where

$$C_1 = \frac{F\,|x_0/f\,(0,x_0)|}{1 - \|\ell\|\,(|x_0/f\,(0,x_0)| + \|h\|_{L^1})}$$

and

$$C_1 = \frac{F}{1 - \|\ell\|\,(|x_0/f(0,x_0)| + \|h\|_{L^1})}.$$

Let

$$w\,(t) = C_1 + C_2 \int_0^t \gamma\,(s)\,\Omega\,(u\,(s))\;ds.$$

Then, $u(t) \le w(t)$ and a direct differentiation of $w(t)$ yields

$$\begin{aligned} &w'\,(t) \le C_2\gamma\,(t)\,\Omega\,(w\,(t))\,, \\ &w\,(0) = C_1, \end{aligned} \tag{4.8}$$

that is,

$$\int_0^t \frac{w'\,(s)}{\Omega\,(w\,(s))}\,ds \le C_2 \int_0^t \gamma\,(s)\;ds \le C_2\,\|\gamma\|_{L^1}\,.$$

A change of variables in the above integral gives that

$$\int_{C_1}^{w(t)} \frac{ds}{\Omega\,(s)} \le C_2\,\|\gamma\|_{L^1} < \int_{C_1}^{\infty} \frac{ds}{\Omega\,(s)}.$$

Now, an application of mean value theorem yields that there is a constant $M > 0$, such that $w(t) \le M$, for all $t \in J$. This further implies that

$$|u\,(t)| \le w\,(t) \le M, \qquad \text{for all } t \in J.$$

Thus, Conclusion (ii) of Theorem 2.2 does not hold. Therefore, the operator inclusion $x \in AxBx$ and consequently, FDI (4.1) has a solution on $J$. This completes the proof. ∎

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